

# Bi-Lipschitz equivalent Alexandrov surfaces, I

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## 1 Basic definitions and statements

Recently M. Bonk and U. Lang [BL] proved, that if a complete Riemannian manifold  $M$  is homeomorphic to the plane  $\mathbb{R}^2$  and satisfies the conditions  $\int_M K^+ dS < 2\pi$ ,  $\int_M K^- dS < \infty$ , then Lipschitz distance  $d_L(M, \mathbb{R}^2)$  between  $M$  and  $\mathbb{R}^2$  satisfies the inequality

$$d_L(M, \mathbb{R}^2) \leq \ln \left( \frac{2\pi + \int_M K^- dS}{2\pi - \int_M K^+ dS} \right)^{\frac{1}{2}}.$$

This inequality is sharp if curvature does not change its sign. Here  $K^+ = \max\{K, 0\}$ ,  $K^- = \max\{-K, 0\}$ , and  $dS$  is the area element. We will remind definition of Lipschitz metric a little later.

In fact, this result was obtained in [BL] for the class of Alexandrov surfaces more wide than the class of Riemannian manifolds.

This paper is inspired by the paper [BL] mentioned above; we will investigate surfaces which are not necessary simply connected. In contrast to the case of surfaces homeomorphic to  $\mathbb{R}^2$ , in more general cases we do not have any standard models. By this reason we try to estimate the Lipschitz distance between two homeomorphic surfaces. Our estimate occurs to be far from the optimal one (and in this part we even restrict ourselves proving finiteness of distances). Our consideration is naturally divided into two parts: asymptotic at infinity and study of compact surfaces.

Our readers supposed to be familiar with the basic notions of two dimensional manifolds of bounded total (integral) curvature theory. Its expositions can be found, for instance, in [AZ] and [Resh].

Hereafter Alexandrov surface means a *complete* two dimensional manifold of bounded curvature with a boundary; the boundary (which may be empty) is supposed to consist of finite number of curves with finite variation of turn.

We introduce the following notations: let  $M$  be an Alexandrov surface with a metric  $d$ ,  $\omega$  be its curvature, which is a signed measure,  $\omega^+$ ,  $\omega^-$  be the positive and negative parts of the curvature, and  $\Omega = \omega^+ + \omega^-$  be the variation of the curvature. For any Riemannian manifold,  $\omega^+ = \int_M K^+ dS$ ,  $\omega^- = \int_M K^- dS$ .

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A point  $p$  carrying curvature  $2\pi$  and a boundary point carrying turn  $\pi$  are called peak points.

Recall that dilatation  $\text{dil } f$  of a Lipschitz map  $f: X \rightarrow Y$ , where  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces, is defined by the equality

$$\text{dil } f = \sup_{x, x' \in X, x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}.$$

A homeomorphism  $f$  is called bi-Lipschitz if both maps,  $f$  and  $f^{-1}$ , are Lipschitz ones. The impression

$$d_L(X, Y) = \inf_{f: X \rightarrow Y} \ln(\max\{\text{dil}(f), \text{dil}(f^{-1})\})$$

is called Lipschitz distance between  $X$  and  $Y$ ; here infimum is taken over all Lipschitz homeomorphisms  $f: X \rightarrow Y$ . Metric spaces  $X, Y$  are bi-Lipschitz equivalent if and only if  $d_L(X, Y) < \infty$ .

**Theorem 1.** *Let two compact Alexandrov surfaces be homeomorphic one to another, have no peak points. Then these surfaces are bi-Lipschitz equivalent one to another.*

This theorem is trivial for Riemannian manifolds because two homeomorphic smooth 2-manifolds are always diffeomorphic. Theorem 1 does not give an upper estimate for Lipschitz distance via some finite set of geometric characteristics of the surfaces, like their diameters, total curvatures, systolic constants, etc. However such an estimate does exist; we are going to publish this result in a separate paper.

Let  $T$  be an end; i.e., an Alexandrov surface homeomorphic to a closed disk with the removed center and such that for every sequence of points  $p_i \in T$  with images in the disk converging to the center, the condition  $d(a, p_i) \rightarrow \infty$  holds for  $i \rightarrow \infty$ ; here  $a$  is any fixed point. Such a sequence is called diverging (or going to infinity) one.

Let us denote the turn of the end boundary  $\partial T$  by  $\sigma$ . We call the value  $v(T) = -\omega(T) - \sigma$  a growth speed of the end  $T$ . The Cohn-Vossen inequality says that this value is not negative. The growth speed of an end is positive if and only if  $\lim_{i \rightarrow \infty} \frac{l(\gamma_i)}{d(a, p_i)} > 0$ , where  $l(\gamma_i)$  is the length of the shortest noncontractible loop with the vertex  $p_i$ . Under condition  $\Omega(T) < \infty$ , this limit is well-defined and is not greater than 2.

As it was proved in ([Hub], [Ver]), a complete Alexandrov surface satisfying the condition  $\omega^-(M) < \infty$  is homeomorphic to a closed surface with finite number of removed points. Ends are appropriate closed neighborhoods of these points. So, any Alexandrov surface can be cut into a compact part and some ends.

**Theorem 2.** *Let two complete Alexandrov surfaces  $M_1, M_2$  be homeomorphic one to another, satisfy the condition  $\omega^-(M_i) < \infty$  and contain neither points*

with curvature  $2\pi$ , nor boundary points with turn  $\pi$ . Also, let all ends of these surfaces have nonzero growth speed. Then these surfaces are bi-Lipschitz equivalent.

**Remark 1.** We call two ends to be equivalent if their intersection contains an end. It is not difficult to see that equivalent ends have equal growth speeds. This means that the theorem above does not depend on how are cut surfaces into compact parts and ends.

Simple examples show that ends having zero speed (even having no peak points) are not necessary bi-Lipschitz equivalent. For instance, none of the surfaces obtained by rotation of the following graphs are bi-Lipschitz equivalent:

$$\{y = \sqrt{x}, x \geq 1\}, \quad \{y = 1, x \geq 0\}, \quad \{y = e^{-x}, x \geq 0\}.$$

Note that surfaces of rotation obtained from the last graph and the graph of the function  $\{y = e^{-2x}, x \geq 0\}$  are bi-Lipschitz equivalent.

Also note that an end having zero growth speed can not be bi-Lipschitz equivalent to an end with nonzero growth speed.

An end is rotationally symmetric if its isometry group contains a subgroup whose restriction to the boundary is transitive. The problem of bi-Lipschitz equivalence of ends can be reduced to the similar problem for rotationally symmetric ends. More precisely, the following theorem takes place.

**Theorem 3.** *Every end  $T$  satisfying the condition  $\omega^-(T) < \infty$  and having no peak points is bi-Lipschitz equivalent to a rotationally symmetric end with the same growth speed.*

**Remark 2.** If  $v(T) > 0$ , then the end  $T$  contains a smaller end  $T_1$ , such that the Lipschitz distance between  $T_1$  and a plane with a disk of length  $l = l(\partial T_1)$  removed can be estimated from above by a value depending on  $v$  and  $l$  only.

This remark can be proved by a minor modification of arguments from [BL]; so we omit the proof.

We see that the problem of bi-Lipschitz classification of ends with zero growth speed is reduced to a problem for functions in one variable. Intuitively, one can imagine a zero speed end as “having a peak point (with curvature  $2\pi$ ) at infinity”. From this point of view, it is naturally to expect analogy between classification of zero speed ends and bi-Lipschitz classification of neighborhoods of finite peak points. Note that for smooth surfaces with isolated singularities the classification problem for neighborhoods of peak points was considered by D. Grieser [Gr].

#### **Sketch of the proof.**

The idea of our proof of Theorem 1 is simple. Here is the sketch of the proof. Preliminary a simply connected region  $\Delta$  bounded by a simple closed curve with three selected points was called a generalized triangle. These points are vertices of the triangle, intervals of the curve between vertices being its sides. We always assume that lengths of the sides satisfy the triangle inequality, the sides are

geodesic broken lines. Actually we will consider only generalized triangles with small variations of curvature and turn of its sides. We will often omit the word “generalized”.

A partition of a 2-manifold  $M$  is a set of generalized triangles in  $M$ , whose interiors do not overlap and whose union is the whole  $M$ . A triangulation of  $M$  is a partition such that for any two triangles of it their intersection is either a side or a vertex.

We will partition both Alexandrov surfaces  $M_1, M_2$  into generalized triangles in such a way that these triangles have small variations of curvature and turn of its sides, but angles are separated from zero. In particular, we include all points carrying essential portion of curvature in the set of vertices of the triangles. We prove that each triangle of the partition is bi-Lipschitz equivalent to its comparison triangle; i.e., to a planar triangles with the same side lengths; one can choose such a bi-Lipschitz mapping to preserve length of sides. For this we use the method introduced by I. Bakelman [Bak] for constructing Tchebysev coordinates in Alexandrov surfaces. Replacing each generalized triangle of our partition by its comparison triangle and attaching last planar triangles together in according with the same combinatorial scheme, we will get two surfaces  $P_1, P_2$  equipped with polyhedral metrics. In according to our construction, each polyhedron  $P_i$  is bi-Lipschitz equivalent to the surface  $M_i, i = 1, 2$ . Finally, note that the polyhedra  $P_1$  and  $P_2$  are bi-Lipschitz equivalent one to another. This ends the proof.

Theorem 2 follows from Theorem 1 and the lemma below.

**Lemma 1.** *Every end  $T$  with nonzero growth speed, without peak points and with  $\omega^-(T) < \infty$  is bi-Lipschitz equivalent to  $\mathbb{R}^2$  with a disk removed (as always, we suppose that boundary of the end is a curve with finite variation of turn and no peak points on the end and its boundary). Moreover, there is a bi-Lipschitz equivalence inducing an affine map on the boundary of the end.*

In case when curvature and turn of the boundary are not great, the lemma can be proved by a minor modification of arguments from [BL]; to do this it is sufficient to suppose that  $\omega^+(T) + \tau^+(\partial T) < \pi$ , where  $\tau^+(\partial T)$  is the positive turn of the end boundary (from the side of the end).

To prove the lemma in the general case, it is sufficient to cut the end into an annular and an end satisfying the last condition and to apply Theorem 1 to the annular.

Lemma 1 also follows from Theorem 3 and Theorem 1. Indeed, Theorem 3 easily implies that every end with nonzero growth speed is bi-Lipschitz equivalent to a cone over a circle of appropriate length with a round neighborhood of the vertex removed. The last surface is obviously bi-Lipschitz equivalent to the plane with a disk removed.

To prove Theorem 3, we will use again a modification of I. Bakelman’s construction [Bak].

## 2 Triangles having small curvature.

We begin with simple statements about planar triangles. Let  $\triangle ABC$  be a flat triangle. Denote by  $a, b, c$  its sides opposite to the angles  $\angle A, \angle B, \angle C$ , correspondingly. We will denote side lengths by the same letters as sides.

**Lemma 2.** *Let planar triangles  $\triangle ABC$  and  $\triangle \overline{ABC}$  be such that  $b = \overline{b}$ ,  $c = \overline{c}$ ,  $\angle A \leq \angle \overline{A} \leq L \cdot \angle A$ , and  $L\angle \overline{A} - \angle A \leq \pi(L-1)$ , where  $L > 1$ . Then the optimal bi-Lipschitz constant of the affine transformation mapping  $\triangle ABC$  to  $\triangle \overline{ABC}$  is not greater than  $L$ .*

It is sufficient to find eigenvalues of the affine transformation under consideration to prove the lemma. ■

**Corollary 4.** *Let triangles  $\triangle ABC$  and  $\triangle \overline{ABC}$  satisfy the conditions:*

$$L^{-1} \leq c/\overline{c} \leq L, \quad L^{-1} \leq b/\overline{b} \leq L, \\ \epsilon \leq \angle A \leq \pi - \epsilon, \quad \epsilon \leq \angle \overline{A} \leq \pi - \epsilon.$$

*Then the optimal bi-Lipschitz constant of the affine transformation mapping  $\triangle ABC$  on  $\triangle \overline{ABC}$  is not greater than  $(L\frac{\pi}{2\epsilon})^2$ .*

**Lemma 3.** *Let triangles  $\triangle ABC$  and  $\triangle \overline{ABC}$  satisfy the condition:  $b = \overline{b}$ ,  $c = \overline{c}$ ,  $D \in BC$ . Denote by  $\overline{D}$  the image of the point  $D$  under affine transformation mapping  $\triangle ABC$  onto  $\triangle \overline{ABC}$ .*

*Let  $\angle BAD = \alpha$ ,  $\angle CAD = \beta$ ,  $\angle \overline{BAD} = \alpha_1$ ,  $\angle \overline{CAD} = \beta_1$ . Suppose that  $\alpha < \frac{\pi}{2}$ ,  $\beta < \frac{\pi}{2}$  and*

$$0 < \angle \overline{BAC} - \alpha < \frac{\pi}{2}, \quad 0 < \angle \overline{BAC} - \beta < \frac{\pi}{2}.$$

*Then*

$$|\alpha - \alpha_1| \leq |\angle BAC - \angle \overline{BAC}|, \quad |\beta - \beta_1| \leq |\angle BAC - \angle \overline{BAC}|.$$

**Proof.** Suppose that  $\angle \overline{BAC} > \angle BAC$ . Let us show that in this case  $\alpha_1 \geq \alpha$ ,  $\beta_1 \geq \beta$ . After these equalities  $\alpha + \beta = \angle BAC$ ,  $\alpha_1 + \beta_1 = \angle \overline{BAC}$ , will imply the lemma.

Since  $\overline{D}$  is the image of  $D$ ,

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin \alpha_1}{\sin \beta_1}.$$

Suppose that  $\alpha_1 < \alpha$ . Consider the point  $Z$  in  $\overline{BC}$  such that  $\angle \overline{BAZ} = \alpha$ . Then

$$\frac{\sin \alpha}{\sin \beta} < \frac{\sin(\overline{BAZ})}{\sin(\overline{CAZ})},$$

therefore  $\sin \beta > \sin(\angle \overline{BAC} - \alpha)$ . But  $0 < \beta < (\angle \overline{BAC} - \alpha) < \frac{\pi}{2}$ . This means that  $\sin \beta < \sin(\angle \overline{BAC} - \alpha)$ . Contradiction.

The case  $\angle \overline{BAC} < \angle BAC$  can be considered analogously. ■

Now let us go back to Alexandrov surfaces. For a generalized triangle  $\Delta \subset M$ , we denote

$$\Omega'(\Delta) = \Omega(\text{int}(\Delta)) + \sigma(\Delta),$$

where  $\sigma(\Delta)$  is the sum of variations of turn of the triangle sides from inside.

**Lemma 4.** *For every  $\theta > 0$ , there exist numbers  $\delta(\theta) > 0$  and  $L(\theta) > 1$  with the following property: if every angle of a generalized triangle  $\Delta ABC$  is not less than  $\theta$ , and if  $\Omega'(\Delta ABC) < \delta(\theta)$ , then there exists a bi-Lipschitz map of the triangle  $ABC$  on its comparison triangle with the constant  $L(\theta)$ , whose restriction on the boundary of the triangle preserves lengths.*

This lemma looks like to be obvious, and the fact we could find neither an appropriate reference nor a very simple proof a little surprising seems to be very surprising.

**Proof.** 1. It is sufficient to prove our lemma only for polyhedral metrics. So we suppose that our metric is a polyhedral one. We will use a construction which is a minor modification of I. Bakelman's one, see [Bak]. Suppose that our  $\Delta ABC$  (equipped with a polyhedral metric) is a part of a complete surface  $M$  homeomorphic to  $\mathbb{R}^2$  and flat outside the triangle. (For that, let us cut out a comparison triangle  $\Delta \overline{ABC}$  from  $\mathbb{R}^2$  and then attach  $\Delta ABC$  instead of  $\Delta \overline{ABC}$ .) Then we extend sides  $\overline{AB}$  and  $\overline{AC}$  as rays  $\overline{AB}^*$  and  $\overline{AC}^*$ . Denote the sector  $B^* \overline{AC}^*$  by  $S$ .

We will partition the sector  $S$  (more precisely, some region of it containing  $\Delta ABC$ ) onto flat parallelograms coming into contact one with another along the whole sides in such a way that four parallelograms adjust to each vertex (except ones belonging the boundary of  $S$ ). This allows immediately to introduce Tchebysev coordinates in the sector  $S$ . These coordinates give (after some additional deformation straightening the side  $BC$ ) required bi-Lipschitz map.

2. To simplify exposition, suppose that  $\Delta ABC$  is an ordinary (not generalized) triangle with zero turn of its sides. The general case differs from this model by nonessential details only. Note that the difference of corresponding angles of triangles  $\Delta ABC$  and  $\Delta \overline{ABC}$  is not greater than some function of  $\Omega'(\Delta ABC)$  (this function can be given explicitly), which goes to zero together with  $\Omega'$ . The proof of that is standard and based on the theorem about “arc and chord” ([AZ], Lemma 5 of Chapter 9), the Gauss–Bonnet theorem, and comparison theorem for (nongeneralized) triangles.

As a result, we can choose  $\delta$  to be so small (smallness depends on  $\theta$  only), that every angle of the triangle  $\Delta \overline{ABC}$  is not less than  $\frac{2}{3}\theta$ .

3. Let  $\angle \overline{C}$  be the smallest angle of the triangle, and  $\angle \overline{A}$  be the greatest one. Then angles  $\angle \overline{B}$  and  $\angle \overline{C}$  are acute and each of them do not exceed  $\frac{1}{2}\pi - \frac{1}{3}\theta$ .

4. Let us consider the line  $l$  containing the ray  $\overline{AB}^*$  and begin to shift it (continuously) parallel to itself inside the sector  $S$ . Initially this line will cut

a flat “oblique” semi-strip from  $S$ . We will continue this process until vertices of the metric appear in  $l$  for the first time. Denote by  $A_{11}$  the intersection of  $l$  and  $AC^*$ , and let  $A_{12}, \dots, A_{1n_1}$  be vertices of the metric which appears in  $l \cap S$ , numbered “from left to right”.

From every point  $A_{1i}$  we emulate a geodesic  $A_{1i}B_{1i}$  outside the cut off semi-strip and so that  $|A_{1i}B_{1i}| = |A_{1j}B_{1j}|$  for all  $i, j$  and besides  $\angle B_{1i}A_{1i}A_{1,i+1} + \angle A_{1i}A_{1,i+1}B_{1,i+1} = \pi$ . If the length of  $|A_{1i}B_{1i}|$  is sufficiently small, then we get a strip consisting of flat parallelograms  $A_{1i}B_{1i}B_{1,i+1}A_{1,i+1}$ . Let us choose  $|A_{1i}B_{1i}|$  so that the vertexes of the metric appear for the first time on the broken line  $B_{11}B_{12}B_{1,3} \dots$ . Following this process we will get the partition of the sector  $S$  by parallelograms. (the last one in this set of parallelograms is supposed to be an infinite oblique semi-strip) To make parallelograms to be adjacent along the whole sides, it is sufficient to slit some of them into more narrow parallelograms.

5. If  $\delta < \theta$  the process described above can be continued until we exhaust all sector  $S$  and it will produce bijective mapping of the sector  $S$  onto the first quadrant  $S_0$  of the plane with the oblique coordinates  $Ouv$ , where the coordinate angle  $\angle uOv$  equals to the angle  $\angle A = \alpha$  of the sector  $S$ .

Indeed, there could be only one obstacle: namely, at some step the broken line  $\Lambda_k = A_{k1}, A_{k2} \dots$  might touch itself or the ray  $AB^*$ . In both cases we would obtain a closed broken line bounding a region  $G$  homeomorphic to some disk. Applying the Gauss–Bonnet formula to  $G$ , we would come to a contradiction with smallness of the curvature of our triangle (in compare with its angles).

6. Now we can introduce Tchebychev coordinates in  $S$ . For the coordinates  $(u, v)$  of a point  $p$  we'll take the lengths of the broken lines connecting this point with the rays  $AC^*$ ,  $AB^*$ , such that in every parallelogram crossed by these lines the latter ones go along the intervals parallel the corresponding sides of the parallelogram. Let  $S_0$  be the first coordinate quadrant of the plane  $\mathbb{R}^2$ , and  $\varphi: S \rightarrow S_0$  be the coordinate mapping constructed according to this plan.

7. On  $S_0$  we define the metric by the linear element

$$ds^2 = du^2 + 2 \cos \tau(u, v) du dv + dv^2,$$

where

$$\tau(u, v) = \alpha - \omega(\varphi^{-1}(D_{uv}))$$

and  $D_{uv}$  is the parallelogram  $[0 \leq u' < u, 0 \leq v' < v]$ .

This linear element makes  $S_0$  to be a metric space  $(S_0, d_1)$ .

8. The map  $\varphi$  is an isometry of  $S$  to  $(S_0, d_1)$ . It is not difficult to check this consequently along all parallelograms of our partition of  $S$ .

Consider, in addition, the standard flat metric  $d_2$  defined by the linear element  $ds^2 = du^2 + 2 \cos \alpha du dv + dv^2$  on  $S_0$ . The map  $\text{id}: (S_0, d_1) \rightarrow (S_0, d_2)$  is linear on every parallelogram. The inequality  $|\tau(x, y) - \alpha| \leq \Omega'(S) < \delta$  shows that this map is bi-Lipschitz with constant  $(\delta\mu)$ . Here and further we denote by  $\mu$  positive constants (may be different) which depend on  $\theta$  only.

The image of the side  $BC$  of the triangle  $\triangle ABC$  under the map  $\varphi_1 = \text{id} \circ \varphi$  is a broken line  $\Lambda = Y_0Y_1 \dots Y_l$ , where  $Y_0 = B' = \varphi_1B$ ,  $Y_l = C' = \varphi_1C$ . We want to prove, that there is a  $\mu$ -bi-Lipschitz transformation  $\zeta$ , which move the region

$Q$ , bounded by this broken and the shortest  $A'B'$ ,  $A'C'$  (where  $A' = \varphi_1 A$  is the vertex of the sector  $S_0$ ) onto the triangle  $\triangle A'B'C'$ . After that, it will be sufficient to apply Lemma 2 to the flat triangles  $\triangle A'B'C'$  and  $\triangle ABC$  to finish the proof of the lemma.

The transformation  $\zeta$  is defined as follows. Denote by  $X_i$  the intersection of the ray  $A'Y_i$  and the side  $B'C'$  of the triangle  $\triangle A'B'C'$ . Now we map affine each triangle  $\triangle A'Y_iY_{i+1}$  onto the corresponding triangle  $\triangle A'X_iX_{i+1}$ . Let us show that it gives a  $\mu$ -bi-Lipschitz map we need.

Indeed, every interval  $Y_iY_{i+1}$  of the broken  $\Lambda$  is an affine image of an interval located in one of the parallelograms. By our construction, parallelograms in  $(S_0, d_2)$  are disposed by “horizontal” rows. Adding “horizontal” and “vertical” rays we can consider each point  $Y_i$  to be a vertex of a parallelogram. As  $\delta$  is small in compare with  $\theta$ , every interval  $Y_iY_{i+1}$  is a diagonal of the corresponding parallelogram. Let a ray  $N_i$  be the “upper” bound of  $k$ -th row, so that  $Y_i \in N_i$ , and  $\tilde{N}_i = \varphi_1^{-1}(N_i)$ . It is not difficult to see that variation of turn of broken line  $\tilde{N}_i$  is not greater than  $\Omega'(M) < \delta$  (this is a rough estimate). Let us apply the Gauss–Bonnet formula to the region bounded by the shortest  $AB$ , intervals of the shortest  $AC$ ,  $BC$  and the curve  $\tilde{N}_i$ . This gives immediately that the angle between  $\tilde{N}_i$  and the shortest  $BC$  is different from the angle  $\angle B$  not greater than on  $2\delta$ . That means that the angle between  $\tilde{N}_i$  and starting at  $\tilde{N}_i$  edge of the broken line  $\Lambda$ , and so also the angle between the same edge and the ray  $A'B'$ , is differ from the angle  $\angle B'$  of the triangle  $\triangle A'B'C'$  not greater than on  $3\delta$  (that follows easily from Lemma 3). Now it is not difficult to calculate that

$$\left| \frac{A'Y_i}{A'X_i} - 1 \right| < \frac{10\delta}{\alpha}$$

(under the condition, that  $\delta$  is small in comparison with  $\alpha$ ). Applying Lemma 4 to the couples  $\triangle A'X_iX_{i+1}$  and  $\triangle A'Y_iY_{i+1}$ , we get that triangle  $\triangle A'B'C'$  and  $\triangle ABC$  are bi-Lipschitz equivalent with some constant  $\mu$ . Unfortunately the restriction of the map we constructed is not an isometry for the side  $BC$ . Nevertheless this restriction changes distances not greater than in  $\mu$  times; therefore it is possible to correct our map in every triangle  $\triangle A'X_iX_{i+1}$  in such a way that it remains to be a bi-Lipschitz equivalence (with some constant  $\mu'$ ) but becomes to be an isometry on the boundary of the triangle. So we got the required map. ■

**Lemma 5.** *There exists  $\xi > 0$  such that if a generalized triangle  $\triangle ABC$  in Alexandrov space satisfies the conditions:*

$$\angle ABC \geq \frac{\pi}{5}, \angle ACB \geq \frac{\pi}{5}, \angle BAC > 0, \Omega'(\triangle ABC) < \xi,$$

*then there exists a bi-Lipschitz map of the triangle  $\triangle ABC$  onto its comparison triangle whose restriction to the boundary maps vertices to the corresponding vertices and preserves lengths.*



In contrast to the previous lemma, now we allow one of angles to be arbitrary small; but from another side now we can not estimate bi-Lipschitz constant.

**Proof.** To prove our Theorem, we cut our triangle into triangles in such a way, that each of them is bi-Lipschitz equivalent to its comparison triangle. We suppose that  $\alpha = \angle BAC < \frac{1}{100}$ , otherwise Lemma 4 would say that our triangle is  $L(\frac{1}{100})$ -bi-Lipschitz equivalent to its comparison triangle  $\triangle \overline{ABC}$ . Now we choose  $\chi = \min\{\delta(\frac{1}{100}\pi), \frac{1}{1000}\}$ , see Lemma 4.

It is easy to see that there exist points  $A_1, C_1, B_1$  on the sides  $BC, AC$  and  $AB$ , correspondingly, such that

$$|AB_1| = |AC_1|, |BA_1| = |BB_1|, |CA_1| = |CC_1|.$$

Consider shortest (w.r.t. the induced metric of the triangle) connecting these points. They partition  $\triangle ABC$  onto four triangles (see an explanation below). Let us repeat the same construction for the triangle  $\triangle AB_1C_1$ ; i.e., choose points  $A_2, C_2, B_2$  in the sides  $B_1C_1, AC_1$  и  $AB_1$ , such that

$$|AB_2| = |AC_2|, |B_1A_2| = |B_1B_2|, |C_1A_2| = |C_1C_2|.$$

Let us continue this process. It is not difficult to calculate that all the angles of all the triangles we obtained are separated from zero and  $\pi$ , for instance, they lie between  $0,05\pi$  and  $0,95\pi$ . Therefore, according to Lemma 4, all such triangles, except the triangle  $\triangle AB_iC_i$ , are  $L(0,05\pi)$ -bi-Lipschitz equivalent to their comparison triangles.

Note that the partition process of the triangle can be continued up to infinity and  $\sum |B_iC_i| = \sum (|B_iB_{i+1}| + |C_iC_{i+1}|) \leq |AB| + |AC|$ , so that  $|B_iC_i| \rightarrow 0$ , when  $i \rightarrow \infty$ , and  $B_i, C_i \rightarrow A$ , as  $\alpha = \angle BAC > 0$ . It follows that there exists an  $i = i_0$  such that  $\Omega'(\triangle AB_{i_0}C_{i_0}) < \delta(\alpha)$  and hence  $\triangle AB_{i_0}C_{i_0}$  is bi-Lipschitz equivalent to its comparison triangle (again according to Lemma 4).

We complete the proof of the theorem using the backward induction on  $i$ :  $\triangle AB_{i_0}C_{i_0}$  is bi-Lipschitz equivalent to its comparison triangle. Suppose, that the same is true for  $\triangle AB_{i+1}C_{i+1}$ , where  $i \leq i_0$ , and prove for  $\triangle AB_iC_i$ . The latter is cut into 4 triangles each being bi-Lipschitz equivalent to its comparison triangle. Consider the analogous partition for the comparison triangle  $\triangle \overline{AB_iC_i}$ . As the curvature of  $\triangle ABC$  is small being compared with the angles of the triangles under consideration (except  $\angle BAC$ ), triangles of the partition of the triangle  $\triangle \overline{AB_iC_i}$  are almost the same as comparison triangles for the corresponding triangles of our partition of  $\triangle AB_iC_i$ . That is why it is easy to construct bi-Lipschitz mappings for these couples of triangles. With this the proof is completed. ■

### 3 Compact Alexandrov surfaces

To prove Theorem 1 we need the following

**Lemma 6.** *For any  $\xi > 0$  there exists a partition by generalize triangles of every compact Alexandrov surface  $M$  without peak points such that for any triangle  $\triangle$  of this partition has positive angles and  $\Omega'(\triangle) < \xi$ .*

**Proof.** First of all, we triangulate a small neighborhood of each point  $p$  such that  $\Omega(p) \leq \frac{1}{2}\xi$ , where  $\xi$  was chosen according to lemma 5. We choose such neighborhoods to be bounded by broken lines and not overlapping. It is easy to triangulate these neighborhoods to satisfy conditions of the lemma.

To partition the remain part of the surface  $M$  we use Theorems 2 from Chapter 3 of the book [AZ]: Any compact subset of  $M$  with a polyhedral boundary can be covered by a set of arbitrary small pairwise nonoverlapping simple triangles such that in every triangle there is no side equals to the sum of two other sides.

It is clear that these triangles can be chosen to be so small that for each of them  $\Omega' < \frac{1}{2}\xi$ . All that is left, is to deform these triangles to turn them into generalized triangles to remove the zero angles and not to break the other suppositions of the Lemma. It is sufficient to remove one zero angle and to use induction over the number of zero angles. Suppose that there is at least one zero angle. As we have no peak points, we can find two adjacent angles, say,  $\angle BOA$  and  $\angle AOC$ , the first one being equal to zero and the second one being nonzero. Note, that the curve  $AOC$  may either consist of two sides or be a whole side of a generalized triangle. Now it is sufficient to replace a very short initial interval of the curve  $OA$  by a two-component broken line  $ODE$ , where  $E \in OA$ , lying in the sector  $\angle AOC$ , by a very small but nonzero angle with  $OA$  and such that variation of the turn of the broken line  $ODEA$  exceeds variation of the turn of the shortest line  $OA$  very slightly. ■

#### **Proof of Theorem 1.**

Let us consider a partition of a compact Alexandrov surface  $M$  satisfying Lemma 6. We want to prove that every triangle of this partition is bi-Lipschitz equivalent to a flat triangle with the same side lengths. Besides, corresponding bi-Lipschitz maps can be chosen so that their restrictions to the boundaries of the triangles have to be isometries. For a triangle with two angles not less than  $\frac{1}{5}\pi$  each, this is true by Lemma 5. Therefore we can assume that the triangle  $\triangle$  under consideration has two angles, each of which is less than  $\frac{1}{5}\pi$ . Then the third angle is not less than  $\frac{3}{5}\pi - \Omega'(\triangle) \geq \frac{3}{5}\pi - \frac{1}{500}$ . Let  $\triangle = \triangle ABC$ , the angle  $\angle B$  being the greatest one.

As the curvature of the triangle  $\triangle$  is small, the angles of any lune formed by two shortest sides and containing in  $\triangle$  do not exceed  $\frac{1}{500}$ . Therefore there is a point  $X \in AC$  such that any shortest  $BX$  makes angles with  $AB$  and  $CB$  not less than  $\frac{3}{10}\pi - \frac{1}{500}$ . Some simple calculation shows that the both triangles,  $\triangle ABX$  and  $\triangle CBX$ , satisfy the conditions of Lemma 5. Hence, both of the triangles are bi-Lipschitz equivalent to their comparison triangles respectively. It is easy to see that the same is true for  $\triangle ABC$ . Thus  $M$  is bi-Lipschitz equivalent to some surface with a polyhedral metric.

Now it remains to use the fact that if two polyhedral surfaces  $X$  and  $Y$  are homeomorphic, then there exists a piecewise linear homeomorphism  $g: X \rightarrow Y$  between them. Then according to [RS] we can find such isomorphic subdivisions  $X'$  and  $Y'$  of these triangulations (by flat triangles) that  $g$  is linear on every triangle of  $g$  and transform it into corresponding triangle of  $Y'$ . Taking

this into account, we derive that two homeomorphic polyhedral Alexandrov spaces are bi-Lipschitz equivalent, and hence the same is valid for two arbitrary homeomorphic Alexandrov spaces satisfying the conditions of the theorem. ■

## 4 Rotationally symmetric ends

An end  $T$  is rotationally symmetric if there is an isometry group of  $T$  acting transitively in  $\partial T$ .

### Proof of Theorem 3.

If an end has nonzero speed, then Remark 2 says that it is bi-Lipschitz equivalent to  $\mathbb{R}^2$  with a disk removed. We have mentioned already, that this fact can be proved by a minor modification of the method of the paper [BL]. Therefore we can suppose that the end  $T$  has zero speed. In addition, it is sufficient to prove the theorem for polyhedral ends having a finite number of vertices. And we can restrict ourselves only with ends satisfying the following conditions:

- 1) the boundary  $\Gamma$  of the end is either a geodesic loop or a polygon, all angles of which being not greater than  $\pi$ .
- 2)  $\Omega(T) + \sigma(\Gamma) < \epsilon = \frac{1}{1000}$ , where  $\sigma(\Gamma)$  is the variation of the boundary turn of the end.

Indeed, we can cut off a tubular neighborhood of the boundary in such a way that the remaining end will satisfy the conditions 1) – 2) and the annulus we cut off will be bi-Lipschitz equivalent to a flat annulus (Theorem 1).

Further we assume the conditions 1) – 2) to be fulfilled. Subsequent proof is similar to our proof of Lemma 4. The only difference is that now we will construct a partition of the end into flat trapezoids; these trapezoids will be placed at layers, and all trapezoids in one layer will have equal highs. Each layer will be bi-Lipschitz equivalent to a surface of revolution, bi-Lipschitz constants will be uniformly bounded, and restrictions of corresponding bi-Lipschitz maps to boundaries will preserve lengths.

To do that, from every “angular” point  $X_i$ ,  $i = 1, \dots, m$ , of the geodesic broken  $\Gamma$  (i.e, from points in which  $\Gamma$  has nonzero turn) we emanate a geodesic forming equal angles with branches of  $\Gamma$  starting at  $X_i$ ; i.e., going along the bisector of the angle between the branches. Choose a small number  $h_1 > 0$  and, in every such a geodesic, select a point  $X_{1i}$  such that  $|X_i X_{1i}| = (\cos \alpha_i)^{-1} h_1$ , where  $2\alpha$  is the turn of  $\Gamma$  at  $X_i$ .

Now we connect cyclically the points  $X_{1i}$  by shortest  $X_{1i} X_{1i+1}$ . For sufficiently small  $t_1$  the quadrangles  $X_i X_{1i} X_{1i+1} X_{i+1}$  are flat nonoverlapping trapezoids having highs  $h_1$  and angles  $\alpha_i$  and  $\alpha_{i+1}$  adjoined to the “bottom” base. (If  $\Gamma$  is a loop with zero turn everywhere except the vertex, we get one trapezoid glued with itself along  $X_1 X_{11}$ .)

Now we will increase  $h_1$  until one of the following events happens:

- a) A vertex of the metric appears for the first time in the broken line  $\Gamma_1 = X_{11}, X_{12}, \dots, X_{1m}$ .

b) Shortests  $X_i X_{1i}$ ,  $X_{1i+1} X_{i+1}$  meet together at a point  $X_{1i} X_{1i+1}$  for some  $i$ .

For sure, several such appearances and meetings can happen simultaneously. It is not difficult to see that nothing but events a) – b) can happen as curvature is small.

We repeat the described construction starting with the broken line  $\Gamma_1$ . (The number of its vertices may be more numerous just as less numerous than the number of vertices of the broken line  $\Gamma$ .) We get the second row of trapezoids, and so on. Just as in the proof of Lemma 4, our supposition guarantees that this process will not stop until all the vertices of the metric are exhausted. In addition, the turn of the broken line  $\Gamma_k$  is the sum of the turn of the broken line  $\Gamma_{k-1}$  and curvatures of the additional vertices, so that  $\sigma(\Gamma_k) \leq \Omega(T) + \sigma(\Gamma)$ . In particular, all the angles of constructed trapezoids differ from  $\pi/2$  not more than  $\frac{1}{1000}$ .

Denote the length of the broken line  $\Gamma_k$ ,  $k = 0, 1, \dots$  by  $L_k = L(\Gamma_k)$ . Here we presume  $\Gamma = \Gamma_0$ . Now we consider an annular  $C_i$  lying on the cone with the total angle  $\theta = t_k^{-1}(L_k - L_{k-1})$  around its vertex  $O$  and bounded by circles of radii  $\theta^{-1}L_k$  and  $\theta^{-1}L_{k+1}$  centered at  $O$ . The lengths of these circles are  $L_k$  and  $L_{k+1}$ , and they distanced one from another by  $h_k$ . (If  $L_k = L_{k+1}$ , then the cone degenerates into a cylinder, simplifying our considerations).

Our next (and final) aim is to show that every annual layer between  $\Gamma_k$  and  $\Gamma_{k+1}$  can be bi-Lipschitz mapped onto  $C_k$ , the constant depending only on  $\epsilon$ , we have chosen. The number  $k$  of the layer is supposed to be fixed and further it will be omitted in notations. Put  $a_i = |X_{ki} X_{k+1i}|$ ,  $b_i = |X_{k+1i} X_{k+1i+1}|$ ,  $a = \sum a_i$ ,  $b = \sum b_i$ . Consider the trapezoid  $ABCD$  with  $|AD| = a$ ,  $|BC| = b$ ,  $\angle DAB = \pi$  and mark points  $A = A_0, A_1, \dots, A_m = D$  and  $B = B_0, B_1, \dots, B_m = C$  on its bases such that  $|A_i A_{i+1}| = a_i$ ,  $|B_i B_{i+1}| = b_i$ . Simple calculations show that angles  $\beta_i$  between the intervals  $A_i B_i$  and the trapezoid bases are uniformly separated from zero and  $\pi$  by the constant depending only on  $\epsilon$ . In fact, as we have  $|a_j - b_j| = h |\operatorname{tg} \alpha_j + \operatorname{tg} \alpha_{j+1}| \leq 2h(|\alpha_j| + |\alpha_{j+1}|)$ , (the last inequality holds because  $\epsilon$  is small), then

$$|\operatorname{ctg} \beta_i| = h^{-1} |\sum_{j=1}^{i+1} (a_j - b_j)| \leq 4 \sum_{j=1}^m |\alpha_j| \leq \epsilon.$$

Now it is not difficult to construct a map of each trapezoid of the end partition onto corresponding trapezoid  $A_i B_i A_{i+1} B_{i+1}$  and hence a map of each annual layer onto corresponding trapezoid  $ABCD$ . At last, it is easy to map the trapezoid  $ABCD$  onto the corresponding annual  $C_i$  on the cone. (All the maps can be constructed in such a way that their restrictions on the boundaries preserve the lengths).

■

### Final Remarks: Alexandrov polyhedra.

Together with Alexandrov surfaces it is possible to consider polyhedra glued of them.

Alexandrov polyhedron is a connected 2-polyhedron  $P$  glued from a finite number of Alexandrov surfaces under the conditions (i) the gluing are made along the boundary curves, and (ii) parts attached one with another have equal

lengths (more precisely, the gluing is made along isometries of boundaries).

A particular case of Alexandrov polyhedra are 2-polyhedra of the curvature bounded above investigated in [BB]. Notions of essential edge and maximal face used there are of pure topological nature and hence can be applied in our case. Each maximal face is an Alexandrov surface (by Alexandrov's Gluing Theorem). We do not exclude a "boundary curve" consisting of one point (the turn of such a curve is, by definition,  $2\pi - \theta$ , where  $\theta$  is the total angle around this point). It might be also possible to allow existence of edges without any faces adjacent to them. Nevertheless we will not do it here. Also for simplicity we restrict ourselves by polyhedra without boundary edges.

**Remark.** This definition of Alexandrov polyhedra is of constructive character. We would like to find an axiomatic (workable) definition of Alexandrov polyhedra. But alas, we could not manage to do it.

For Alexandrov polyhedra, curvature can be naturally defined. In fact, it is sufficient to define it for essential vertices and on subsets of essential edges, as out of these vertices and edges curvature of a set is assumed to be equal to its curvature as a part of the maximal face. If  $\gamma$  is an essential edge,  $e \subset \gamma$ , and  $\tau_i$  is the turn of  $\gamma$  as of the part of the boundary of the  $i$ -th face adjacent to  $\gamma$ , then by definition  $\omega(e) = \sum_i \tau_i(e)$ , where sum is spread over all faces adjacent to the edge  $\gamma$ . For the vertex  $p$ , curvature is defined by the formula

$$\omega(p) = (2 - \chi(\Sigma_p))\pi - s(\Sigma_p),$$

where  $\chi$  and  $s$  are Euler characteristic and the length of the graph  $\Sigma_p$ . Under this definition of the curvature, for compact Alexandrov polyhedra Gauss–Bonnet Theorem is true in its ordinary form.

The situation with positive and negative parts of the curvature is more complicate. We also need to make the definition of ends more precise. Having in mind the straightforward generalizations of the theorem above, we must exclude analogies of peak points and the ends with zero speed growth. The former means that we should suppose that the maximal faces having to contain neither inner points with the curvature  $2\pi$ , nor boundary points for which the boundary has the turn  $\pi$ . This restriction can be described more strictly using the language of curvatures of the essential edges and vertices. Namely, we need to suppose that there are no points  $p$  in which  $\max_i \tau_i^+(p) = \pi$  on the essential edges. Here maximum is taken over all the faces adjacent to  $\gamma$ . For the vertex  $p$ , let us consider its link  $\Lambda(p)$ .

Vertices of a link correspond to essential edges of the polyhedra (starting at  $p$ ) and its edges correspond to the maximal faces to which these edges are adjacent; in particular, the edges of a link can be circles (do not containing essential vertices or containing only one vertex). The space of directions at the point  $p$  naturally induces a semi-metric on  $\Lambda(p)$ . Now let us assume this semi-metric to be a metric; i.e.,  $\Lambda(p)$  to be homeomorphic to the space of directions. In other words, we claim that every edge of a link has nonzero length in the angle metric.

Now about ends. Let  $Q$  be a part of a polyhedron with complete infinite metric homeomorphic to the direct product of a finite graph and  $\mathbb{R}^+$ . It is reasonable to consider a cone; i.e., single-point compactification  $\bar{Q} = Q \cup \{p\}$  of the space  $Q$ , so that all paths going to the point  $p$  have infinite length.

We will think of ends as of subcones of this cone which are cones over the arcs of the graph connecting its two neighboring essential vertices. Thus an end can look either as  $S^1 \times \mathbb{R}^+$  or as  $[a, b] \times R^+$ .

With these definitions in mind the theorems proved above can be easily extended from surfaces to the Alexandrov polyhedra.

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